More on FT

Lecture 10

4CT.5

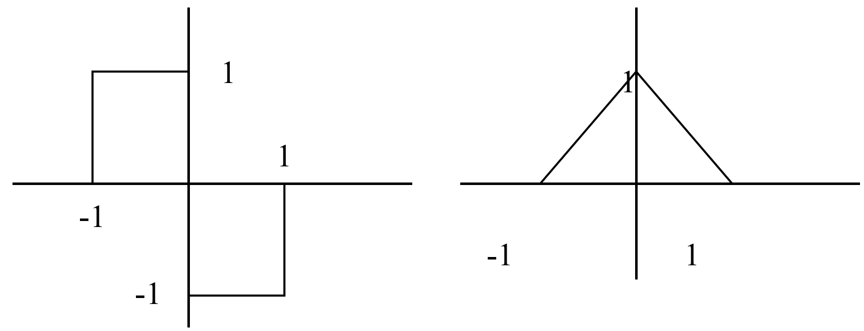
3CT.3-5,7,8

Homework

- Problem (1)
  - Show that $F(j\omega)$ of an even function $f(t)=f(-t)$ is real
  - Show that $F(j\omega)$ of an odd function $f(t)=-f(-t)$ is imaginary
- Problem (2)
  - Find $F(j\omega)$ for the following functions and sketch their amplitude and phase spectra:
    - $f_a(t)=e^{-at}\, u(t)$;
    - $f_b(t)= e^{at}\, u(-t)+e^{-at}\, u(t)$;
    - $f_c(t)= -e^{at}\, u(-t)+e^{-at}\, u(t)$;
    - $f_d(t)= e^{-t} \sin 10t\, u(-t)$

Homework

- Problem (3)
  - Calculate the $F(j\omega)$ for the waveforms below. Note that the second is the integral of the first.



- 4CT.5.1, 4CT.5.2
- 3CT.5.1, 3CT.5.2, 3CT.5.3

Homework Answers #1

- Problem (1)
  - Show that *F(jω)* of an even function *f(t)=f(-t)* is real
  - Show that *F(jω)* of an odd function *f(t)=-f(-t)* is imaginary

$$F(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt=\int_{-\infty}^{0} f(t)e^{-j\omega t}dt+\int_{0}^{\infty} f(t)e^{-j\omega t}dt$$

let $x=-t$, then $dx=-dt$

$$=-\int_{\infty}^{0} f(-x)e^{j\omega x}dx+\int_{0}^{\infty} f(t)e^{-j\omega t}dt$$

$$=\int_{0}^{\infty} f(x)e^{j\omega x}dx+\int_{0}^{\infty} f(t)e^{-j\omega t}dt$$

$$=\int_{0}^{\infty} f(t)[\cos\omega t+j\sin\omega t]dt+\int_{0}^{\infty} f(t)[\cos\omega t-j\sin\omega t]dt$$

$$=2\int_{0}^{\infty} f(t)\cos\omega t dt \Rightarrow real!$$

$$F(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt=\int_{-\infty}^{0} f(t)e^{-j\omega t}dt+\int_{0}^{\infty} f(t)e^{-j\omega t}dt$$

let $x=-t$, then $dx=-dt$

$$=-\int_{\infty}^{0} f(-x)e^{j\omega x}dx+\int_{0}^{\infty} f(t)e^{-j\omega t}dt$$

$$=\int_{0}^{\infty} f(-x)e^{j\omega x}dx+\int_{0}^{\infty} f(t)e^{-j\omega t}dt$$

$$=-\int_{0}^{\infty} f(x)e^{j\omega x}dx+\int_{0}^{\infty} f(t)e^{-j\omega t}dt$$

$$=-\int_{0}^{\infty} f(t)[\cos\omega t+j\sin\omega t]dt+\int_{0}^{\infty} f(t)[\cos\omega t-j\sin\omega t]dt$$

$$=-2j\int_{0}^{\infty} f(t)\sin\omega t dt \Rightarrow imaginery!$$

Homework Answers #2

- Problem (2)
  - Find $F(j\omega)$ for the following functions and sketch their amplitude and phase spectra:
    - $f_a(t)=e^{-at}\,u(t);$
    - $f_b(t)= e^{at}\,u(-t)+e^{-at}\,u(t);$
    - $f_c(t)= -e^{at}\,u(-t)+e^{-at}\,u(t);$
    - $f_d(t)= e^{-t}\sin 10t\;u(-t)$

$$a)\ \Im[e^{-\alpha t}u(t)] = \int_{-\infty}^{\infty} e^{-\alpha t}u(t)e^{-j\omega t}dt$$

$$= \int_{0}^{\infty} e^{-(\alpha+j\omega)t}dt = \frac{e^{-(\alpha+j\omega)t}}{-(\alpha+j\omega)}\Big|_0^\infty$$

$$= 0 - \frac{1}{-(\alpha+j\omega)} = \frac{1}{(\alpha+j\omega)}$$

OR

$$\Im[u(t)] = \frac{1}{j\omega}$$

$$\Im[e^{-\alpha t}u(t)] = \frac{1}{\alpha+j\omega};\ \text{due to frequency displacement}$$

$$b)\ \Im[e^{\alpha t}u(-t)] = \int_{-\infty}^{\infty} e^{\alpha t}u(-t)e^{-j\omega t}dt$$

$$x=-t; dx=-dt$$

$$= \int_{\infty}^{-\infty} -e^{-\alpha x}u(x)e^{+j\omega x}dx$$

$$= \int_{-\infty}^{\infty} e^{-\alpha x}u(x)e^{+j\omega x}dx$$

$$= \int_{0}^{\infty} e^{-\alpha x}e^{+j\omega x}dx$$

$$= \frac{e^{-(\alpha-j\omega)x}}{-(\alpha-j\omega)}\Big|_0^\infty = 0 - \frac{1}{-(\alpha-j\omega)} = \frac{1}{(\alpha-j\omega)}$$

OR

$$\Im[f(t)] = F(j\omega)$$

$$\Im[f(-t)] = \int_{-\infty}^{\infty} f(-t)e^{-j\omega t}dt;\ \text{let } -t=x, dt=-dx$$

$$= -\int_{\infty}^{-\infty} f(x)e^{j\omega x}dx = \int_{-\infty}^{\infty} f(x)e^{j\omega x}dx = F(-j\omega)$$

$$\Im[e^{\alpha t}u(-t)] = \frac{1}{\alpha-j\omega}$$

$$\Im[e^{\alpha t}u(-t)+e^{-\alpha t}u(t)] = \frac{1}{(\alpha-j\omega)} + \frac{1}{(\alpha+j\omega)}$$

$$= \frac{\alpha+j\omega}{(\alpha^2+\omega^2)} + \frac{\alpha-j\omega}{(\alpha^2+\omega^2)} = \frac{2\alpha}{(\alpha^2+\omega^2)}$$

Homework Answers #3

- Problem (2)
  - Find $F(j\omega)$ for the following functions and sketch their amplitude and phase spectra:
    - $f_a(t)=e^{-at}\,u(t);$
    - $f_b(t)= e^{at}\,u(-t)+e^{-at}\,u(t);$
    - $f_c(t)= -e^{at}\,u(-t)+e^{-at}\,u(t);$
    - $f_d(t)= e^{t}\sin 10t\,u(-t)$

$$c)\ \Im[-e^{\alpha t}u(-t)+e^{-\alpha t}u(t)] = -\frac{1}{(\alpha - j\omega)}+\frac{1}{(\alpha + j\omega)}$$

$$= -\frac{\alpha + j\omega}{(\alpha^2+\omega^2)}+\frac{\alpha - j\omega}{(\alpha^2+\omega^2)} = \frac{-2j\omega}{(\alpha^2+\omega^2)}$$

$$d)\ \Im[e^{t}u(-t)] = \frac{1}{(1-j\omega)}$$

$$\Im[f(t)\sin 10t] = \Im[f(t)\frac{(e^{j10t}-e^{-j10t})}{2j}] = \frac{1}{2j}\{F[j(\omega-10)]-F[j(\omega+10)]\}$$

$$\Im[e^{t}\sin 10t u(-t)] = \frac{1}{2j}\{\frac{1}{1-j(\omega-10)}-\frac{1}{1-j(\omega+10)}\}$$

$$= \frac{1}{2j}\{\frac{1-j(\omega+10)-[1-j(\omega-10)]}{[1-j(\omega-10)][1-j(\omega+10)]}\}$$

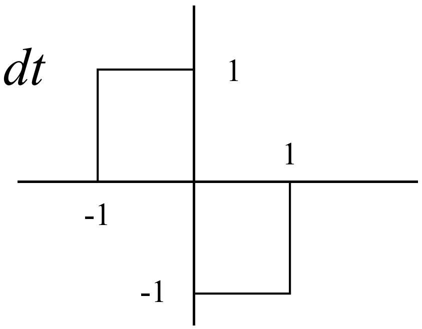
$$= \frac{1}{2j}\{\frac{-20j}{1-j(\omega-10)-j(\omega+10)-(\omega-10)(\omega+10)}\}$$

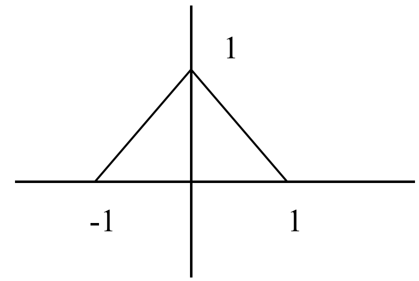
$$= \frac{-10}{1-[(\omega-10)(\omega+10)]-2j\omega}$$

$$= \frac{-10}{101-\omega^2-2j\omega}$$

Homework Answers #4

- Problem (3)
  - Calculate the *F(jω)* for the waveforms below. Note that the second is the integral of the first.





$$a)\ F_a(j\omega) = \Im[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$= \int_{-1}^{0} e^{-j\omega t}dt - \int_{0}^{1} e^{-j\omega t}dt$$

$$= \frac{1}{-j\omega}\{e^{-j\omega t}|_{-1}^{0} - e^{-j\omega t}|_{0}^{1}\}$$

$$= \frac{1}{-j\omega}\{(1-e^{j\omega}) - (e^{-j\omega}-1)\}$$

$$= \frac{1}{-j\omega}\{(2-[e^{j\omega}+e^{-j\omega}]\}$$

$$= \frac{2}{-j\omega}\{1-\cos\omega\}$$

$$b)\ F_b(j\omega) = \frac{F_a(j\omega)}{j\omega}$$

$$= \frac{2}{\omega^2}\{1-\cos\omega\}$$

4CT.5.1

$$X(F) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}dt$$

$$\text{If } F = 0, X(0) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi 0t}dt = \int_{-\infty}^{\infty} x(t)1dt = \int_{-\infty}^{\infty} x(t)dt$$

4CT.5.2

$$x(t) = \int_{-\infty}^{\infty} X(F)e^{j2\pi Ft}dF$$

$$\text{If } t = 0, x(0) = \int_{-\infty}^{\infty} X(F)e^{-j2\pi 0F}dF = \int_{-\infty}^{\infty} X(F)1dF = \int_{-\infty}^{\infty} X(F)dF$$

3CT.5.1

$$h(t) = \int_{-\infty}^{\infty} H(F) e^{j2\pi Ft} dF$$

$$X(F) = \delta(F)$$

$$h(t) = \int_{-\infty}^{\infty} \delta(F) e^{j2\pi Ft} dF = 1$$

$$y(t) = \int_{-\infty}^{\infty} 1x(\tau) d\tau = \int_{-\infty}^{\infty} x(t-\tau) d\tau$$

The system is unstable since if $x(t)$ is bounded the area under may the infinite.

The system is not causal since $h(t) \neq 0$ for $t < 0$.

3CT.5.2

$$H(F) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi Ft} dt$$

$$h(t) = \delta(t)$$

$$H(F) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi Ft} dt = 1$$

$$y(t) = \int_{-\infty}^{\infty} \delta(\tau) x(t-\tau) d\tau = x(t)$$

The system is all pass filter and is causal.

3CT.5.3

$$h(t) = \int_{-\infty}^{\infty} H(F) e^{j2\pi Ft} dF$$

$$X(F) = \frac{1}{2}\delta(F-100) + \frac{1}{2}\delta(F+100)$$

$$h(t) = \int_{-\infty}^{\infty} [\frac{1}{2}\delta(F-100) + \frac{1}{2}\delta(F+100)] e^{j2\pi Ft} dF$$

$$= \frac{1}{2}[\int_{-\infty}^{\infty} \delta(F-100) e^{j2\pi Ft} dF + \int_{-\infty}^{\infty} \delta(F+100) e^{j2\pi Ft} dF]$$

$$= \frac{1}{2}(e^{j2\pi 100t} + e^{-j2\pi 100t}) = \cos(2\pi 100t)$$

$$y(t) = \int h(\tau) x(t-\tau) d\tau = \int \cos(2\pi 100\tau) x(t-\tau) d\tau$$

This system is unstable and not causal since $h(t) \neq 0$ for $t < 0$.